

## OPEN BOOKS AND EXACT SYMPLECTIC COBORDISMS

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ABSTRACT. Given two open books with equal pages we show the existence of an exact symplectic cobordism whose negative end equals the disjoint union of the contact manifolds associated to the given open books, and whose positive end induces the contact manifold associated to the open book with the same page and monodromy given by the concatenation of the given ones. Given a contact 3-manifold, according to Eliashberg, there is a symplectic cobordism to a fibration over the circle with symplectic fibres. We extend this result to higher dimensions recovering a result by Dörner-Geiges-Zehmisch. Our cobordisms can also be thought of as the result of the attachment of a generalised symplectic 1-handle.

## INTRODUCTION

Let  $\Sigma$  denote a compact,  $2n$ -dimensional manifold admitting an exact symplectic form  $\omega = d\beta$  and let  $Y$  denote the Liouville vector field defined by  $i_Y\omega = \beta$ . Suppose that  $Y$  is transverse to the boundary  $\partial\Sigma$ , pointing outwards. These properties are precisely the ones requested for  $\Sigma$  to be a page of an abstract open book in the contact setting. Given a symplectomorphism  $\phi$  of  $(\Sigma, \omega)$ , equal to the identity near  $\partial\Sigma$ , one can, following a construction of Thurston and Winkelnkemper [12] or rather its adaption to higher dimensions by Giroux [9] respectively, associate a  $(2n+1)$ -dimensional contact manifold  $M_{(\Sigma, \omega, \phi)}$  to the set of data  $(\Sigma, \omega, \phi)$ .

The main result of the present paper is part of the author's thesis [11].

**Theorem 1.** *Given two symplectomorphisms  $\phi_0$  and  $\phi_1$  of  $(\Sigma, \omega)$ , equal to the identity near the boundary  $\partial\Sigma$ , there is an exact symplectic cobordism whose negative end equals the disjoint union of the contact manifolds  $M_{(\Sigma, \omega, \phi_0)}$  and  $M_{(\Sigma, \omega, \phi_1)}$ , and whose positive end equals  $M_{(\Sigma, \omega, \phi_0 \circ \phi_1)}$ . If, in addition, the page  $(\Sigma, \omega)$  is a Weinstein manifold, then so is the cobordism.*

For  $n = 2$  the above statement is due to Baker–Etnyre–van Horn-Morris [2] and, independently, Baldwin [3]. The general case of Theorem 1 was independently obtained by Avdek in [1], where the cobordism is associated with a so called *Liouville connected sum*. In [11] I observed that the cobordism in Theorem 1 can also be understood as result of the attachment of a generalized symplectic 1-handle of the form  $D^1 \times N(\Sigma)$ , where  $N(\Sigma)$  denotes a vertically invariant neighbourhood of the symplectic hypersurface  $\Sigma$ . I will shed some more light on this in §4. From the methods introduced in the proof of Theorem 1 we can deduce some further applications, such as the strong fillability of contact manifolds associated with *symmetric open books*, a certain class of fibre bundles over the circle (cf. §3), and the result which we describe next.

Let  $(M, \xi)$  be a closed, oriented,  $(2n + 1)$ -dimensional contact manifold supported by an open book with page  $(\Sigma, \omega)$  and monodromy  $\phi$ . Suppose further that  $(\Sigma, \omega)$  symplectically embeds into a second  $2n$ -dimensional (not necessarily closed) symplectic manifold  $(\Sigma', \omega')$ , i.e.

$$(\Sigma, \omega) \subset (\Sigma', \omega').$$

Let  $M'$  be the symplectic fibration over the circle with fibre  $(\Sigma', \omega')$  and monodromy equal to  $\phi$  over  $\Sigma \subset \Sigma'$  and equal to the identity elsewhere.

**Theorem 2.** *There is a cobordism  $W$  with  $\partial W = (-M) \sqcup M'$  and a symplectic form  $\Omega$  on  $W$  for which  $(M, \xi)$  is a concave boundary component, and  $\Omega$  induces  $\omega'$  on the fibres of the fibration  $M' \rightarrow S^1$ .*

For  $n = 2$  we could, for example, choose  $\Sigma'$  to be the closed surface obtained by capping off the boundary components of  $\Sigma$ . Then Theorem 2 would recover one of the main results (Theorem 1.1) in [4]. The low-dimensional case ( $n = 2$ ) of Theorem 2 was, using different methods, already carried out in [14]. Theorem 2 has previously been proved by Dörner, Geiges and Zehmisch and will appear in [8]. The proof in the present paper uses slightly different methods. One may think of Theorem 2 as an extension of the result in [4], or [14] respectively, to higher dimensions.

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## 1. PRELIMINARIES

**1.1. Symplectic cobordisms.** Suppose we are given a symplectic  $2n$ -manifold  $(X, \omega)$ , oriented by the volume form  $\omega^n$ , such that the oriented boundary  $\partial X$  decomposes as  $\partial X = (-M_-) \sqcup M_+$ , where  $-M_-$  stands for  $M_-$  with reversed orientation. Suppose further that in a neighbourhood of  $\partial X$  there is a Liouville vector field  $Y$  for  $\omega$ , transverse to the boundary and pointing outwards along  $M_+$ , inwards along  $M_-$ . The 1-form  $\alpha = i_Y \omega$  restricts to  $TM_{\pm}$  as a contact form defining cooriented contact structures  $\xi_{\pm}$ .

We call  $(X, \omega)$  a **(strong) symplectic cobordism** from  $(M_-, \xi_-)$  to  $(M_+, \xi_+)$ , with **convex** boundary  $M_+$  and **concave** boundary  $M_-$ . In case  $(M_-, \xi_-)$  is empty  $(X, \omega)$  is called a **(strong) symplectic filling** of  $(M_+, \xi_+)$ . If the Liouville vector field is defined not only in a neighbourhood of  $\partial X$  but everywhere on  $X$  we call the cobordism or the filling respectively **exact**.

A **Stein manifold** is an affine complex manifold, i.e. a complex manifold that admits a proper holomorphic embedding into  $\mathbb{C}^N$  for some large integer  $N$ . By work of Grauert [10] a complex manifold  $(X, J)$  is Stein if and only if it admits an exhausting plurisubharmonic function  $\rho: X \rightarrow \mathbb{R}$ . Eliashberg and Gromov's symplectic counterparts of Stein manifolds are *Weinstein manifolds*.

A **Weinstein manifold** is a quadruple  $(X, \omega, Z, \varphi)$ , see [6], where  $(X, \omega)$  is an exact symplectic manifold,  $Z$  is a complete globally defined Liouville vector field, and  $\varphi: X \rightarrow \mathbb{R}$  is an exhausting (i.e. proper and bounded below) Morse function

for which  $Z$  is gradient-like. Suppose  $(X, \omega)$  is an exact symplectic cobordism with boundary  $\partial X = (-M_-) \sqcup M_+$  and with Liouville vector field  $Z$ . We call  $(X, \omega)$  **Weinstein cobordism** if there exists a Morse function  $\varphi: X \rightarrow \mathbb{R}$  which is constant on  $M_-$  and on  $M_+$ , has no boundary critical points, and for which  $Z$  is gradient-like.

**1.2. Open books.** An **open book decomposition** of an  $n$ -dimensional manifold  $M$  is a pair  $(B, \pi)$ , where  $B$  is a co-dimension 2 submanifold in  $M$ , called the **binding** of the open book and

$\pi: M \setminus B \rightarrow S^1$  is a (smooth, locally trivial) fibration such that each fibre  $\pi^{-1}(\varphi)$ ,  $\varphi \in S^1$ , corresponds to the interior of a compact hypersurface  $\Sigma_\varphi \subset M$  with  $\partial \Sigma_\varphi = B$ . The hypersurfaces  $\Sigma_\varphi$ ,  $\varphi \in S^1$ , are called the **pages** of the open book.

In some cases we are not interested in the exact position of the binding or the pages of an open book decomposition inside the ambient space. Therefore, given an open book decomposition  $(B, \pi)$  of an  $n$ -manifold  $M$ , we could ask for the relevant data to remodel the ambient space  $M$  and its underlying open books structure  $(B, \pi)$ , say up to diffeomorphism. This leads us to the following notion.

An **abstract open books** is a pair  $(\Sigma, \phi)$ , where  $\Sigma$  is a compact hypersurface with non-empty boundary  $\partial \Sigma$ , called the **page** and  $\phi: \Sigma \rightarrow \Sigma$  is a diffeomorphism equal to the identity near  $\partial \Sigma$ , called the **monodromy** of the open book. Let  $\Sigma(\phi)$  denote the mapping torus of  $\phi$ , that is, the quotient space obtained from  $\Sigma \times [0, 1]$  by identifying  $(x, 1)$  with  $(\phi(x), 0)$  for each  $x \in \Sigma$ . Then the pair  $(\Sigma, \phi)$  determines a closed manifold  $M_{(\Sigma, \phi)}$  defined by

$$(1) \quad M_{(\Sigma, \phi)} := \Sigma(\phi) \cup_{\text{id}} (\partial \Sigma \times D^2),$$

where we identify  $\partial \Sigma(\phi) = \partial \Sigma \times S^1$  with  $\partial(\partial \Sigma \times D^2)$  using the identity map. Let  $B \subset M_{(\Sigma, \phi)}$  denote the embedded submanifold  $\partial \Sigma \times \{0\}$ . Then we can define a fibration  $\pi: M_{(\Sigma, \phi)} \setminus B \rightarrow S^1$  by

$$\left. \begin{array}{l} [x, \varphi] \\ [\theta, re^{i\pi\varphi}] \end{array} \right\} \mapsto [\varphi],$$

where we understand  $M_{(\Sigma, \phi)} \setminus B$  as decomposed in (1) and  $[x, \varphi] \in \Sigma(\phi)$  or  $[\theta, re^{i\pi\varphi}] \in \partial \Sigma \times D^2 \subset \partial \Sigma \times \mathbb{C}$  respectively. Clearly  $(B, \pi)$  defines an open book decomposition of  $M_{(\Sigma, \phi)}$ .

On the other hand, an open book decomposition  $(B, \pi)$  of some  $n$ -manifold  $M$  defines an abstract open book as follows: identify a neighbourhood of  $B$  with  $B \times D^2$  such that  $B = B \times \{0\}$  and such that the fibration on this neighbourhood is given by the angular coordinate,  $\varphi$  say, on the  $D^2$ -factor. We can define a 1-form  $\alpha$  on the complement  $M \setminus (B \times D^2)$  by pulling back  $d\varphi$  under the fibration  $\pi$ , where this time we understand  $\varphi$  as the coordinate on the target space of  $\pi$ . The vector field  $\partial\varphi$  on  $\partial(M \setminus (B \times D^2))$  extends to a nowhere vanishing vector field  $X$  which we normalise by demanding it to satisfy  $\alpha(X) = 1$ . Let  $\phi$  denote the time-1 map of the flow of  $X$ . Then the pair  $(\Sigma, \phi)$ , with  $\Sigma = \pi^{-1}(0)$ , defines an abstract open book such that  $M_{(\Sigma, \phi)}$  is diffeomorphic to  $M$ .

**1.3. Compatibility.** A positive contact structure  $\xi = \ker \alpha$  and an open book decomposition  $(B, \pi)$  of an  $(2n + 1)$ -dimensional manifold  $M$  are said to be **compatible**, if the 2-form  $d\alpha$  induces a symplectic form on the interior  $\pi^{-1}(\varphi)$  of each

page, defining its positive orientation, and the 1-form  $\alpha$  induces a positive contact form on  $B$ .

The relevant data to remodel the open book and its compatible contact structure is the triple  $(\Sigma, \omega, \phi)$ , where  $\omega$  denotes the exact symplectic form induced by  $d\alpha$  and  $(\Sigma, \phi)$  is the abstract open book as defined in the previous subsection. Note that in this case the monodromy  $\phi$  defines a symplectomorphism of  $(\Sigma, \omega)$ .

## 2. PROOFS OF THE MAIN THEOREMS

**2.1. Proof of Theroem 1.** Let  $(r, x)$  denote coordinates on a collar neighbourhood  $(-\varepsilon, 0] \times \partial\Sigma$  induced by the negative flow corresponding to the Liouville vector field  $Y$ . Let  $\varrho: \Sigma \rightarrow [0, \infty]$  be a  $C^\infty$ -function on  $\Sigma$  satisfying the following properties:

- $\varrho \equiv 0$  over  $\Sigma \setminus ((-\varepsilon, 0] \times \partial\Sigma)$ ,
- $\varrho \equiv \infty$  over  $\partial\Sigma$ ,
- $\frac{\partial \varrho}{\partial r} > 0$  and  $\frac{\partial \varrho}{\partial x} \equiv 0$  over  $((-\varepsilon, 0] \times \partial\Sigma)$  with coordinates  $(r, x)$ .

Note that over the collar neighbourhood  $(-\varepsilon, 0] \times \partial\Sigma$  the vector field  $Y$  is gradient-like for  $\varrho$ . Consider the space  $\Sigma \times \mathbb{R}^2$  with coordinates  $(p; t, z)$ . This space is symplectic with symplectic form

$$\Omega = \omega + dz \wedge dt.$$

Consider the vector field  $Z'$  on  $\Sigma \times \mathbb{R}^2$  defined by

$$Z' = Y + X,$$

where  $X = (1 - f'(t))z \partial_z + f(t) \partial_t$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  is the function satisfying the following properties:

- $f(\pm\sqrt{2}) = f(0) = 0$ ,
- $|f'(t)| < 1$  for each  $t \in \mathbb{R}$ , and
- $f'$  has exactly two zeros  $t_\pm$  who satisfy  $0 < \pm t_\pm < \sqrt{2}$  and  $f'(t_\pm) > 0$ .

An easy computation shows that  $X$  is a Liouville vector field on  $(\mathbb{R}^2, dz \wedge dt)$  for any function  $f$ . Hence  $Z'$  defines a Liouville vector field on  $(\Sigma \times \mathbb{R}^2, \Omega)$ .

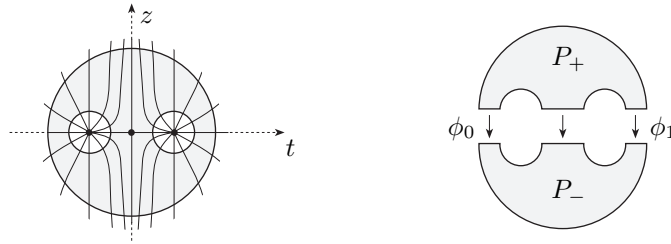


FIGURE 1. Left: Flow lines of the Liouville vector field  $X$ . Right: Construction of  $P(\phi_0, \phi_1)$

We are now ready to define the desired symplectic cobordism  $W$ . Let  $P$  denote the subset of  $\Sigma \times \mathbb{R}^2$  defined by

$$P := \{(p; t, z): \varrho \leq 0, z^2 + t^2 \leq C \text{ and } (t \pm \sqrt{2})^2 + z^2 \geq 1\},$$

where  $C \in \mathbb{R}$  is some constant satisfying  $C > (\sqrt{2} + 1)^2$ . We will now cut  $P$  along  $\{z = 0\}$  and then reglue with respect to  $\phi_0$  and  $\phi_1$  as follows. Set  $P_{\pm} := P \cap \{\pm z \geq 0\}$  and  $P_0 = P \cap \{z = 0\}$ . Obviously  $P_0$  can be understood as part of the boundary of  $P_+$  as well as of  $P_-$ . Now consider

$$P(\phi_0, \phi_1) := (P_+ \sqcup P_-) / \sim_{\Phi},$$

where we identify with respect to the map  $\Phi: P_0 \rightarrow P_0$  (understanding the domain of definition of  $\Phi$  as part of the boundary of  $P_+$  and the target space as part of  $P_-$ ) given by

$$\Phi(p; t, 0) := \begin{cases} (\phi_0(p); t, 0) & , \text{ for } t \leq \sqrt{2} - 1, \\ (\phi_1^{-1}(p); t, 0) & , \text{ for } t \geq \sqrt{2} + 1, \\ (p; t, 0) & , \text{ for } |t| \leq \sqrt{2} - 1. \end{cases}$$

Note that, since  $\phi_0$  and  $\phi_1$  are symplectomorphisms of  $(\Sigma, \omega)$  and  $\Phi$  keeps the  $t$ -coordinates fixed  $\Omega$  descends to a symplectic form on  $P(\phi_0, \phi_1)$  which we will continue to denote by  $\Omega$ . We are now going to define a Liouville vector field  $Z$  on  $P(\phi_0, \phi_1)$ . Let  $g, h: [-\varepsilon, 0] \rightarrow \mathbb{R}$  be the functions satisfying the following properties:

- $g(z) = 0$ , for  $z \in [-\varepsilon, 0]$  near  $-\varepsilon$ ,
- $g(z) = 1$ , for  $z \in [-\varepsilon, 0]$  near  $0$ ,
- $g'(z) \geq 0$ , for each  $z \in [-\varepsilon, 0]$ ,
- $h(z) = g(-\varepsilon - z)$ , for each  $z \in [-\varepsilon, 0]$ ,
- $g(z) + h(z) = 1$ , for each  $z \in [-\varepsilon, 0]$ .

Observe that for these functions we have  $dh = -dg$ . The symplectomorphisms  $\phi_0$  and  $\phi_1$  can be chosen to be exact (cf. [7]), i.e. for  $i = 0, 1$  the equation  $\phi_i^* \beta - \beta = d\varphi_i$  defines a function  $\varphi_i$  on  $\Sigma$ , unique up to adding a constant. By the compactness of  $\Sigma$  we may assume that  $\varphi_0$  takes only negative values whereas  $\varphi_1$  takes only positive values — this will be needed to assure that  $Z$  will be transverse to  $\partial W$ . To avoid confusing indices we will write

$$\Phi^* \beta - \beta = d\varphi$$

to summarise these facts. Over  $P_-$  we define  $Z$  to be given as

$$Z = \left( g(z) (T\Phi^{-1})(Y) + h(z) Y \right) + X - g'(z) \varphi(p) \partial_t.$$

To show that  $Z$  is indeed a Liouville vector field we have to take a look at the Lie derivative of  $\Omega$  along  $Z$ . With the help of the Cartan formula we compute

$$\begin{aligned} \mathcal{L}_Z \Omega &= d(g \Phi^* \beta + h \beta) + dz \wedge dt + d(g' \varphi dz) \\ &= (dg \wedge (\Phi^* \beta) + dh \wedge \beta + g(\Phi^* \omega) + h \omega) + dz \wedge dt + g' d\varphi \wedge dz \\ &= (g' dz \wedge (\Phi^* \beta) - g' dz \wedge \beta + (g + h) \omega) + dz \wedge dt + g' d\varphi \wedge dz \\ &= (g' dz \wedge d\varphi + \omega) + dz \wedge dt - g' dz \wedge d\varphi \\ &= \omega + dz \wedge dt \\ &= \Omega. \end{aligned}$$

Observe that we can extend  $Z$  over  $P_+$  by  $Z'$ . In particular  $Z$  descends to a vector field on  $P(\phi_0, \phi_1)$ . Set

$$W' := \{(p, z, t): \varrho^2 + z^2 + t^2 \leq C \text{ and } \varrho^2 + z^2 + (t \pm \sqrt{2})^2 \geq 1\}$$

and note that we have  $P \subset W'$ . Finally we define the symplectic cobordism  $W$  by

$$W := (W' \setminus P) \cup P(\phi_0, \phi_1).$$

The boundary of  $W$  decomposes as  $\partial W = \partial_- W \sqcup \partial_+ W$ , where we have

$$\partial_- W = \{\varrho^2 + z^2 + (t \pm \sqrt{2})^2 = 1\} \quad \text{and} \quad \partial_+ W = \{\varrho^2 + z^2 + t^2 = C\}.$$

We do not have to worry about the well-definedness of the function  $\varrho$  on  $P(\phi_0, \phi_1) \subset W$  since  $\phi_0$  and  $\phi_1$  can be assumed to equal the identity over  $(-\varepsilon, 0] \times \partial\Sigma$ , which is the only region where  $\varrho$  is non-trivial. Observe that the Liouville vector field  $Z$  is transverse to  $\partial W$  pointing inwards along  $\partial_- W$  and outwards along  $\partial_+ W$ . Finally observe that we indeed have  $\partial_- W = M_{(\Sigma, \omega, \phi_0)} \sqcup M_{(\Sigma, \omega, \phi_1)}$  and  $\partial_+ W = M_{(\Sigma, \omega, \phi_0 \circ \phi_1)}$ , which completes the proof.  $\square$

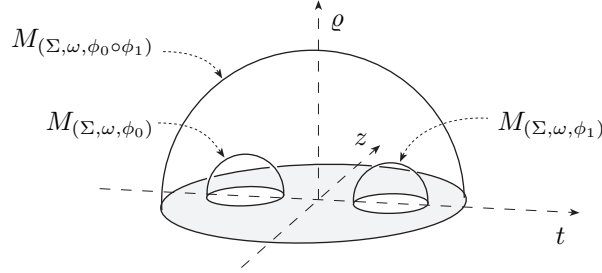


FIGURE 2. Schematic picture of the symplectic cobordism constructed in Theorem 1.

**2.2. Proof of Theorem 2.** Let  $\varrho: \Sigma \rightarrow [0, \infty]$  be a  $C^\infty$ -function on  $\Sigma$  as in the proof of Theorem 1. We can extend this function by  $\infty$  over the rest of  $\Sigma'$ . Analogous to the proof of Theorem 1 we consider the symplectic space  $\Sigma' \times \mathbb{R}^2$  with symplectic form  $\Omega = \omega' + dt \wedge dz$ . Over  $\Sigma \times \mathbb{R}^2 \subset \Sigma' \times \mathbb{R}^2$  we define a Liouville vector field  $Z' = Y + z \partial_z + 2t \partial_t$ . Let  $A$  denote the subset of  $\Sigma' \times \mathbb{R}^2$  defined by

$$A := \{(p, t, z): \varrho \leq 0, z^2 + t^2 \geq 1 \text{ and } z^2 + t^2 \leq 2\}.$$

In analogy of the definition of  $P(\phi_0, \phi_1)$  in the proof of Theorem 1 we define  $A(\phi)$ : set  $A_\pm := A \cap \{\pm z \geq 0\}$  and  $A_0 = A \cap \{z = 0\}$ . We can understand  $A_0$  as part of the boundary of  $A_+$  as well as of  $A_-$ . We define

$$A(\phi) := (A_+ \sqcup A_-) / \sim,$$

where we identify with respect to the map  $\Phi: A_0 \rightarrow A_0$  given by

$$\Phi(p; t, 0) := \begin{cases} (\phi(p); t, 0) & , \text{ for } t < 0, \\ (p; t, 0) & , \text{ for } t > 0. \end{cases}$$

Set

$$W' := \{(p, t, z): \varrho^2 + z^2 + t^2 \geq 1 \text{ and } z^2 + t^2 \leq 2\}$$

and note that we have  $A \subset W'$ . Finally we define the symplectic cobordism  $W$  by

$$W := (W' \setminus A) \cup A(\phi).$$

Observe that  $\Omega$  descends to a symplectic form on  $W$ . Furthermore we indeed have  $\partial W = (-M) \cup M'$ . With  $g, h, \varphi$  as in the proof of Theorem 1 we define a Liouville vector field  $Z$  on  $W_{\varrho \leq 2}$  by

$$Z = (g(z)(T\Phi^{-1})(Y) + h(z)Y) + (z\partial_z + 2t\partial_t) - g'(z)\varphi(p)\partial_t.$$

This vector field is transverse to the lower boundary  $\partial_- W = M_{(\Sigma, \omega, \phi)}$  pointing inwards. Finally observe that  $\Omega$  induces  $\omega'$  on the fibres of the fibration  $M' \rightarrow S^1$ .  $\square$

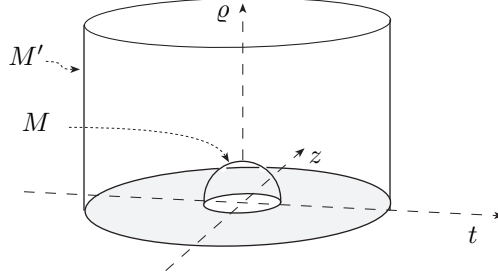


FIGURE 3. Schematic picture of the symplectic cobordism constructed in Corollary 2.

### 3. EXACT FILLINGS OF SYMMETRIC OPEN BOOKS

Let  $(M_0, \xi_0)$  be a contact  $(2n+1)$ -manifold supported by an open book  $(\Sigma, \omega, \phi)$  and let  $(M_1, \xi_1)$  be the contact manifold associated to the open book  $(\Sigma, \omega, \phi^{-1})$ . Denoting by  $B$  the boundary of  $\Sigma$  we can form a new contact manifold  $(M', \xi')$  as follows: the binding  $B$  defines a codimension-2 contact submanifold in  $(M_0, \xi_0)$  as well as in  $(M_1, \xi_1)$ . Their normal bundles  $\nu B_0$  and  $\nu B_1$  admit trivialisations induced by the pages of the respective open book decompositions of  $M_0$  and  $M_1$ . Hence, we can perform the fibre connected sum (cf. [7, §7.4]) along each copy of  $B$  with respect to these trivialisations of the normal bundles and denote the result by  $(M', \xi')$ , i.e. denoting by  $\Psi$  the fibre orientation reversing diffeomorphism of  $B \times D^2 \subset B \times \mathbb{C}$  sending  $(b, z)$  to  $(b, \bar{z})$  we define

$$(M', \xi') := (M_0, \xi_0) \#_{\Psi} (M_1, \xi_1).$$

The result  $(M', \xi')$  defines a fibration over the circle with fibre given by

$$\Sigma' = (-\Sigma) \cup_B \Sigma.$$

Note that each fibre  $\Sigma'$  defines a convex hypersurface, i.e. there is a contact vector field  $X$  on  $(M', \xi')$  which is transverse to the fibres. Furthermore for each fibre  $\Sigma'$  the contact vector field  $X$  is tangent to the contact structure exactly over  $B$ . We will call  $(M', \xi')$  a **symmetric open book**.

**Proposition 3.** *Any symmetric open book admits an exact symplectic filling.*

*Proof.* Consider the symplectic space  $\Sigma \times \mathbb{R} \times [0, 2\pi]$  with coordinates  $(p, t, z)$  and symplectic form  $\Omega = \omega + dz \wedge dt$ . Set

$$W := (\Sigma \times \mathbb{R} \times [0, 2\pi]) / \sim,$$

where we identify with respect to the map  $\Phi: \Sigma \times \mathbb{R} \times \{0\}$  defined by  $(p, t, z) \mapsto (\phi(p), t, z)$ . Since  $\phi$  is a symplectomorphism of  $(\Sigma, \omega)$  the symplectic form  $\Omega$  on  $\Sigma \times \mathbb{R} \times [0, 2\pi]$  descends to a symplectic form on  $W$  which we continue to denote by  $\Omega$ . Note that the fibres of the projection  $W \rightarrow \mathbb{R}$  on the  $\mathbb{R}$ -coordinate are diffeomorphic to the mapping torus  $\Sigma(\phi)$ .

Let  $\varrho: \Sigma \rightarrow [0, \infty]$  be a  $C^\infty$ -function on  $\Sigma$  as in the proof of Theorem 1 and let  $W_C \subset W$ , for some constant  $C > 0$ , denote the subset defined by

$$W_C := \{(p, t, z): \varrho^2 + t^2 \leq C\}.$$

With  $g, h, \varphi$  as in the proof of Theorem 1 we define a Liouville vector field  $Z$  on  $W$  by

$$Z = (g(z)(T\Phi^{-1})(Y) + h(z)Y) + t\partial_t - g'(z)\varphi(p)\partial_t.$$

For sufficiently large  $C > 0$  this vector field is transverse to the boundary  $\partial W_C$  of the subset  $W_C$  pointing outwards. Finally observe that  $\partial W_C = M'$  and that  $Z$  indeed induces the contact structure  $\xi'$ .  $\square$

#### 4. A GENERALIZED SYMPLECTIC 1-HANDLE

We assume that the reader is familiar with the idea behind the symplectic handle constructions due to Eliashberg [5] and Weinstein [13]. For an introduction we point the reader to [7].

Consider  $\mathbb{R} \times \Sigma \times \mathbb{R}$  with coordinates  $(t, p, z)$  and symplectic form  $\Omega = \omega + dz \wedge dt$ . The vector field

$$Z = Y + 2z\partial_z - t\partial_t$$

defines a Liouville vector field for  $\Omega$ . Notice that  $Z$  is gradient like for the function on  $\Sigma \times \mathbb{R}^2$  defined by

$$g(t, p, z) := \varrho^2 + z^2 - \frac{1}{2}t^2,$$

where  $\varrho: \Sigma \rightarrow [0, \infty]$  is a  $C^\infty$ -function on  $\Sigma$  as in the proof of Theorem 1. In particular the Liouville vector field  $Z$  is transverse to the level sets of  $g$  and hence induces contact structures on them. Denote by  $N(\Sigma), N_0(\Sigma) \subset \Sigma \times \mathbb{R}$  the subsets defined by

$$N(\Sigma) := \{\varrho^2 + z^2 \leq 1\} \quad \text{and} \quad N_0(\Sigma) := \{\varrho^2 + z^2 = 0\}.$$

Let  $\mathcal{N} \cong S^0 \times \text{Int}(N(\Sigma))$  and  $\mathcal{N}_0 \cong S^0 \times \text{Int}(N_0(\Sigma))$  denote the set of points  $(t, p, z) \subset g^{-1}(-1)$  which lie on a flow line of  $Z$  through  $S^0 \times \text{Int}(N(\Sigma))$  and  $S^0 \times \text{Int}(N_0(\Sigma))$  respectively. The set  $\mathcal{N}$  is going to play the role of the lower boundary. We now define our **generalized symplectic 1-handle**  $H_\Sigma$  as the locus of points  $(t, p, z) \in \mathbb{R} \times \Sigma \times \mathbb{R}$  satisfying the inequality

$$-1 \leq g(t, p, z) \leq 1$$

and lying on a flow line of  $Z$  through a point of  $\mathcal{N}$ . Since the Liouville vector field  $Y$  is transverse to the level sets of  $g$ , the 1-form

$$\alpha = i_Y \omega + 2z dt + t dz$$

induces a contact structure on the lower and upper boundary of  $H_\Sigma$ .

It is possible to perturb  $H_\Sigma$  without changing the contact structure on the lower and upper boundary as follows. Let  $\nu: \mathbb{R} \times \Sigma \times \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function and let  $H'_\Sigma$  denote the image of  $H_\Sigma$  under the time-1 map of the flow corresponding to



the vector field  $\nu Z$ . Sometimes it is more convenient to work with such a perturbed handle.

**4.1. Attachment and of the handle and its result.** Let  $(M, \xi = \ker \alpha)$  be a  $(2n + 1)$ -dimensional contact manifold. Suppose we are given a strict contact embedding of  $\mathcal{N}$ , endowed with the contact structure induced by  $i_Z \Omega$ , into  $(M, \xi = \ker \alpha)$ . In the following we will describe the symplectic cobordism  $W_{(M, \Sigma)}$  associated to the attachment of the handle  $H_\Sigma$ .

Note that for each point  $x \in \mathcal{N} \setminus \mathcal{N}_0$  there is a point  $\mu(x) > 0$  in time such that the time- $\mu(x)$  map of the flow corresponding to the Liouville vector field  $Z$  maps  $x$  to the upper boundary of  $H_\Sigma$ . This defines a function  $\mu: \mathcal{N} \setminus \mathcal{N}_0 \rightarrow \mathbb{R}^+$  which we may, with respect to the above embedding of  $\mathcal{N}$  in  $(M, \xi = \ker \alpha)$ , extend to a non-vanishing function over  $M \setminus \mathcal{N}_0$ . We continue to denote this map  $M \setminus \mathcal{N}_0 \rightarrow \mathbb{R}^+$  by  $\mu$ . Consider the symplectisation  $(\mathbb{R} \times M, d(e^r \alpha))$  and let  $[0, \mu] \times M$  denote the subset defined by

$$[0, \mu] \times M = \{(r, x) \in \mathbb{R} \times M : 0 \leq r \leq \mu(x)\}.$$

For any point  $(0, x) \in \{0\} \times M \setminus \mathcal{N}_0$  The time- $\mu(x)$  map of the flow corresponding to the Liouville vector field  $\partial_r$  on  $(\mathbb{R} \times M, d(e^r \alpha))$  maps  $(0, x)$  to  $(\mu(x), x)$ . We define  $W_{(M, \Sigma)}$  as follows: start with the disjoint union

$$[0, \mu] \times (M \setminus \mathcal{N}_0) \sqcup H_\Sigma,$$

and define  $W_{(M, \Sigma)}$  as the quotient space obtained by identifying  $(r, x) \in \mathcal{N} \setminus \mathcal{N}_0$  with the image of  $x \in \mathcal{N} \subset H_\Sigma$  under the time- $r$  map of the flow corresponding to the Liouville vector field  $Z$ . This identification does actually respect the symplectic forms (cf. [7, Lemma 5.2.4]) and we indeed end up with an exact symplectic cobordism  $W_{(M, \Sigma)}$ . The concave boundary component  $\partial_- W_{(M, \Sigma)}$  is equal to  $(M, \xi)$  whereas the convex component  $\partial_+ W_{(M, \Sigma)}$  equals

$$(M, \xi) \setminus (S^0 \times \text{Int } N(\Sigma)) \cup_\partial (D^1 \times \partial N(\Sigma), \eta)$$

where  $\eta$  denotes the kernel of the contact form  $i_Y \omega + dt$ .

**4.2. Recovering the 1-handle in the construction of Theorem 1.** Let  $W$  be the cobordism constructed in Theorem 1. A sufficiently small neighbourhood of  $\{t = 0\}$  in  $W$  is given by

$$\{(t, p, z) : \varrho^2 + z^2 + t^2 \leq C\}$$

and we can actually understand this part as embedded in  $\mathbb{R} \times \Sigma \times \mathbb{R}$ . Note that over this neighbourhood the symplectic form  $\Omega = \omega + dz \wedge dt$  and the Liouville vector field  $Z$  from Theorem 1 agree with the symplectic form and the Liouville vector field given in the definition of the generalized symplectic 1-handle  $H_\Sigma$  in §4 above. Further note that we can understand the above neighbourhood of  $\{t = 0\}$  as part of a perturbed handle  $H_\Sigma^\nu$  for a sufficiently chosen  $\nu: \mathbb{R} \times \Sigma \times \mathbb{R} \rightarrow \mathbb{R}$ .

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